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THE FLOW OF LIQUID DOWN AN INCLINED PLANE AT HIGH REYNOLDS NUMBERS*

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The stability of the flow of a layer of incompressible liquid with a free surface down an inclined plane under the force of gravity is investigated for the case of large Reynolds and Froude numbers. The amplitudes of the perturbations which lead to a non-linear problem are found. Problems with initial data are formulated, as well as the boundary value problems with conditions on a moving wall. It is shown that four characteristic zones appear in the field of flow in a transverse direction, changing successively from one to the next. It is noted that the proposed scheme enables one to study detached flows with recirculation zones. The scheme constructed here resembles in many ways the pattern of flow past a plate on which a boundary layer is developed with selfinduced pressure /1-4/.

1. Let a layer of incompressible viscous liquid flow down an inclined plane, making an angle θ with the horizontal, under the force of gravity directed vertically downwards. We shall assume that the unperturbed motion is steady-state motion, with velocity parallel to the inclined plane. We shall choose, as dimensional quantities, the parameters of the unperturbed motion: the velocity of the free boundary U_0, the height of the liquid layer H_0 and the density of the liquid ρ_0. Using them we introduce dimensionless dependent and independent variables. We shall use a Cartesian system of coordinates with the x' axis directed

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along the inclined plane and y' axis directed into the liquid, and we shall measure the y' coordinate from the free surface. The Navier-Stokes equations will be used as the basic equations, and the passage to dimensionless form described here will lead to two parameters, namely the Reynolds number $R = {}^2/_3 V_0 H_0/\nu_0$, where ν_0 is the coefficient of kinematic viscosity and the Froude number $F = U_0/\sqrt{H_0 g_0}$ where g_0 is the acceleration due to gravity.

We specify the conditions of adhesion on the inclined plane for $y' = 1$

$$u = u_w, \quad v = v_w \tag{1.1}$$

where u and v are the components of the velocity vector along the x' and y' axes, and u_w and v_w are the components of velocity of the free surface. Although the major part of the inclined plane is stationary and impermeable and $u_w = v_w = 0$ on it, nevertheless a small part of it may be occupied by a vibrator and we may have there suction or injection, in which case $u_w^2 + v_w^2 \neq 0$.

Let us specify, on the perturbed free surface $y' = \eta$, the equality of the normal and tangential stresses, which can be written, for the approximations discussed below, in the form

$$\begin{gathered}\frac{\partial u}{\partial y'} + \frac{\partial v}{\partial x'} = 0, \quad S_T = \frac{\sigma_0}{\rho_0 H_0 U_0^2} \\ -p + \frac{4}{3}\frac{1}{R}\frac{\partial v}{\partial y'} + S_T \frac{\partial^2 \eta}{\partial x'^2} = 0\end{gathered} \tag{1.2}$$

and the kinematic condition

$$\frac{\partial \eta}{\partial t'} + u \frac{\partial \eta}{\partial x'} = v \tag{1.3}$$

In formulas (1.2) and (1.3) t' is the time, p is the pressure and σ_0 is the dimensional surface tension. The dimensionless parameters R and S_T can be connected with each other using the angle of inclination θ and a new dimensionless parameter γ which can be defined in terms of the dimensional parameters of the liquid only

$$S_T = 4\cdot 3^{-5/3}\gamma R^{-5/3} \sin^{-1/3}\theta, \quad \gamma = \sigma_0 (\rho_0 g_0 \nu_0^{4/3})^{-1} \tag{1.4}$$

Within the approximations discussed below $R \to \infty$, therefore, according to (1.4), $S_T \to 0$, and we can neglect the effect of the surface tension. The term S_T however will be left in (1.2) in order to deal with the case $S_T \sim 1$ as $R \to \infty$ and carry out a formal **parametric** analysis of the problem. In this case the quantity γ must be of the order of $R^{5/3} \sin^{1/3}\theta$. In the case of, for example, mercury (Hg), we have $S_T = {}^8/_{15}$ when $\theta = 10°$ and $R = 820$.

The basic unperturbed flow governed by the Navier-Stokes equations, homogeneous conditions in the plane (1.1) and conditions on the free surface, can be described as follows:

$$\begin{gathered}u_s = U(y') = 1 - y'^2, \quad v_s = 0 \\ p_s = R^{-1}P(y') = F^{-2} y' \cos\theta = {}^4/_3 R^{-1} y' \operatorname{ctg}\theta \\ \eta = 0, \quad U_0 = {}^1/_2 H_0^2 g_0 \sin\theta/\nu_0\end{gathered} \tag{1.5}$$

The stability of the solution of (1.5) in its linear approximation was discussed in /5, 6/. Further analysis of the flows close to the unperturbed flow is based on the non-linear theory. Thus soliton solutions were constructed and three-dimensional effects were taken into account in /7-10/. All these papers dealt with the unperturbed flows at small Reynolds and Froude numbers. The main purpose of the present paper is to study the solutions of the Navier-Stokes equations satisfying conditions (1.1)-(1.3), with the longitudinal velocity representing almost everywhere the perturbation $U(y')$ as $R \to \infty$ and $\theta \neq 0$. The solutions satisfy the additional requirement that, when their amplitudes tend to zero, which ensures the global linearization of the perturbations in the longitudinal velocity relative to $U(y')$ over the whole region, the problem with homogeneous conditions (1.1), will have a solution describing neutral oscillations. From the latter it follows that the relation connecting k with ω will represent, in the case of neutral oscillations, the asymptotic forms, as $R \to \infty$, of one of the neutral stability curves for the corresponding problem for the Orr-Sommerfeld equation.

Let us carry out an asymptotic analysis of the problem, as $R \to \infty$, which will be analogous in many respects to that carried out in /1-3/ when investigating stationary detached flows which was then extended to the non-stationary solutions in /4/. We shall bring into our discussion the main region with transverse dimension $y' = O(1)$, occupying almost the whole flow except narrow sections adjacent to the surface of the inclined plane and the free surface. We shall assume that the characteristic longitudinal dimension $\Delta x'$ in the main region and in all other regions introduced below, is the same. We shall also assume that it depends on the value of R, since $\Delta x' \sim R^{\alpha_x}$ where α_x is a number. The characteristic dimensionless time will also be assumed to be the same and proportional to R^{α_t} in all regions. The relationship connecting the velocity and pressure perturbations with the Reynolds number will however be different in different regions.

Let us write the expansions in the main region in the form (numbering the regions consecutively from the free surface downwards, we shall denote the functions sought and the transverse coordinate in the main region by the subscript 3, and leave the time and longitudinal coordinate, which remain the same in all regions, without any indices)

$$t' = R^{\alpha_t}t, \quad x' = R^{\alpha_x}x, \quad y' = y_3 \tag{1.6}$$
$$u = U(y_3) + R^{\alpha_u}u_{31} + \ldots, \quad v = R^{\alpha_v}v_{31} + \ldots$$
$$p = R^{-1}P(y_3) + R^{\alpha_p}p_{31}$$

Here and henceforth the quantities with indices lm will be functions of t, x, y_l ($l = 1, 2, 3, 4$; $m = 1, 2$) unless otherwise indicated. We can also impose the following constraints based on the physical considerations $\alpha_u, \alpha_v, \alpha_p$ accompanying the Reynolds numbers in (1.6)

$$\alpha_u < 0, \ \alpha_v < 0, \ \alpha_p < 0 \tag{1.7}$$

Experiments show that when the Reynolds number increases, the long-wavelength, low-frequency perturbations become unstable. Therefore we shall assume that

$$\alpha_x > 0, \quad \alpha_t > 0 \tag{1.8}$$

Let us substitute the expansions (1.6) into the system of Navier-Stokes equations. Retaining both terms in the equation of continuity, we obtain the following serial equality and equation

$$\alpha_u - \alpha_x = \alpha_v, \quad \partial u_{31}/\partial x + \partial v_{31}/\partial y_3 = 0 \tag{1.9}$$

In analysing the projections of the equations of conservation of momentum, we shall make the following two assumptions: $\alpha_x > \alpha_t$ and $\alpha_x < 1$. The first assumption is equivalent to the requirement that the perturbation with the subscript 31 should be quasistationary, and the second assumption excludes the influence of viscous stresses on the formation of perturbations. As a result we obtain a single equation (which is identical with (1.9)) and two equations

$$\alpha_v - \alpha_x = \alpha_p \tag{1.10}$$
$$U\frac{\partial u_{31}}{\partial x} + \frac{dU}{dy_3}v_{31} = 0, \quad U\frac{\partial v_{31}}{\partial x} = -\frac{\partial p_{31}}{\partial y_3}$$

The system of three Eqs.(1.9) and (1.10) can be integrated analytically, **and its** solution, satisfying the condition of decay as $x \to -\infty$, has the form

$$u_{31} = \frac{dU}{dy_3}A_{31}, \quad v_{31} = -U\frac{\partial A_{31}}{\partial x}, \quad p_{31} = P_{31} + \frac{\partial^2 A_{31}}{\partial x^2}\int_0^{y_3} U^2\,dy \tag{1.11}$$

Here A_{31} and P_{31} are arbitrary functions of t, x, and they approach zero as $x \to -\infty$.

2. The solution (1.11) was obtained from a system which did not contain viscous stresses, therefore it cannot be used in the problem with homogeneous conditions to help to satisfy the conditions of adhesion. Further, when $y_3 \to 1$, the asymptotic expansion in the function u is violated since the principal term $U(y_3) \to 0$ and $u_{31} \not\to 0$. This forces us to introduce a new boundary region in which the solution must, on the one hand, satisfy the conditions of adhesion at the wall (1.1) and reduce, on the other hand, **to the** solution in the main region.

Let us write the function u from the main region 3, as $y_3 \to 1$, in the form

$$u = 2(1 - y_3) - 2R^{\alpha_u}A_{13}(t, x) + \ldots \tag{2.1}$$

Comparing the orders of magnitude of the first and second term in (2.1) we can say that the asmyptotic expansion in powers of R becomes invalid when $1 - y_3 \sim R^{\alpha_u}$. Using this, we introduce the characteristic transverse scale in the new region, and a new transverse variable

$$y_3 = 1 - R^{\alpha_u}y_4 \tag{2.2}$$

The preliminary stage of matching at the level of exponents enables us to write the asymptotic expansions in the region adjacent to the wall, in the form

$$u = R^{\alpha_u}u_{41} + \ldots, \quad v = R^{\alpha_v+\alpha_u}v_{41} + \ldots \tag{2.3}$$
$$p = R^{-1}P(y_3) + R^{\alpha_p}p_{41} + \ldots$$

The complete matching at the level of the functions, leads to the following limiting conditions as $y_4 \to \infty$:

$$u_{41} \to 2y_4 - 2A_{31} + \ldots, \quad v_{41} \to -2y_4\frac{\partial A_{31}}{\partial x} + \ldots \tag{2.4}$$

$$p_{41} \to P_{31} + \frac{8}{15}\frac{\partial^2 A_{31}}{\partial x^2} + \ldots$$

Let us make an assumption about the nature of the flow in the region next to the wall. First we shall assume that the flow is non-stationary, i.e. in addition to convective terms we shall retain the term containing the time derivative. We shall also assume that the flow is formed under the action of pressure forces and a tangential viscous stress. The last condition is necessary from the mathematical point of view, as it enables us to retain the second derivative in y_4 of u_{41} in the equation of conservation of momentum, which in turn enables us to satisfy the conditions of adhesion. Substituting now the expansions (2.2) and (2.3) into the system of Navier-Stokes equations, we obtain three serial relations and a system of three equations

$$\alpha_u - \alpha_t = 2\alpha_u - \alpha_x, \quad 2\alpha_u - \alpha_x = \alpha_p - \alpha_x \qquad (2.5)$$
$$2\alpha_u - \alpha_x = -\alpha_u - 1$$

$$\frac{\partial u_{41}}{\partial x} + \frac{\partial v_{41}}{\partial y_4} = 0, \quad \frac{\partial p_{41}}{\partial y_4} = 0 \qquad (2.6)$$
$$\frac{\partial u_{41}}{\partial t} + u_{41}\frac{\partial u_{41}}{\partial x} + v_{41}\frac{\partial u_{41}}{\partial y_4} = -\frac{\partial p_{41}}{\partial x} + \frac{2}{3}\frac{\partial^2 u_{41}}{\partial y_4^2}$$

Combining the series relations (1.7), (1.8) and (2.5) and solving the resulting system, we obtain

$$\alpha_x = {}^1/_7, \quad \alpha_t = {}^3/_7, \quad \alpha_u = -{}^2/_7, \quad \alpha_v = -{}^3/_7, \quad \alpha_p = -{}^4/_7 \qquad (2.7)$$

Relations (2.7) imply that the exponents obtained satisfy the necessary physical requirements (1.7) and (1.8).

Eqs.(2.6) represent a well-known Prandtl system for an incompressible, non-stationary boundary layer. In addition to the limiting conditions (2.4), its solution must obey the conditions at the rigid wall, i.e. conditions (1.1) written in terms of the region adjacent to the wall

$$u_{41}(t, x, y_{4w}) = u_{41w}(t, x), \quad v_{41}(t, x, y_{4w}) = v_{41w}(t, x) \qquad (2.8)$$

where $y_{41} = y_{4w}(t, x)$ is the equation of the surface of the rigid wall, as well as the initial conditions with respect to time

$$u_{41}(0, x, y_4) = u_{410}(x, y_4) \qquad (2.9)$$

In a classical formulation of the problem for a boundary layer, the pressure p_{41} is assumed to be a given function. In the formulation given here, on the other hand, in /1-4/, the pressure must be determined in the course of solving the problem. It is for this reason that, unlike in the classical formulation, we have here the limiting conditions (2.4) although only the first and third of these conditions will be independent. As regards the second condition, it follows from the first condition and the equation of continuity.

We find however, that the problem for region 4 adjacent to the wall is not closed, since we have two arbitrary functions P_{31}, A_{31} in the limiting condition (2.4). An analogous situation arises when Poiseuille flow in a plane channel /11/ is analysed. In order to establish the relation between P_{31} and A_{31}, we must consider the regions lying above the main region and adjacent to the free surface.

3. Having found in Sect.2 the exponents of the Reynolds number (2.7), we shall introduce for convenience a small parameter $\varepsilon = R^{-1/7}$ are express in terms of its powers the orders of the functions sought and of the independent variables.

Let us take into account in the unknown functions in the main region (1.6), the following terms of the expansions:

$$u = v(y_3) + \varepsilon^2 u_{31} + \varepsilon^4 u_{32} + \ldots, \quad v = \varepsilon^3 v_{31} + \varepsilon^5 v_{32} + \ldots \qquad (3.1)$$
$$p = \varepsilon^7 P(y_3) + \varepsilon^4 p_{31} + \varepsilon^6 p_{32} + \ldots$$

The powers of ε preceding the functions with the subscript 32 are chosen so that the system of equations governing these function will be inhomogeneous. The complete solutions for the re-introduced functions are very bulky, and therefore we shall only give their asymptotic expressions as $y_3 \to 0$

$$u_{32} = -(P_{31} + A_{31}^2) - y_3\left(2A_{32} + \frac{\partial^2 A_{31}}{\partial x^2} - 2\int_{-\infty}^{x} \frac{\partial A_{31}}{\partial t} dx_1\right) + \ldots \qquad (3.2)$$
$$v_{32} = -\frac{\partial A_{32}}{\partial x} + y_3 \frac{\partial (P_{31} + A_{31}^2)}{\partial x} + \ldots$$
$$p_{32} = P_{32} - y_3\left(\frac{\partial^2 A_{32}}{\partial x^2} - \frac{\partial^2 A_{31}}{\partial x \partial t}\right) + \ldots$$

where $A_{32} = A_{32}(t, x)$, $P_{32} = P_{32}(t, x)$ are arbitrary functions.

Using formulas (1.4), (1.9) and (3.2), we shall write the asmptotic expansions of the functions in the main region (3.1) as $y_3 \to 0$, as follows:

$$u = 1 - y_3^2 + 2\varepsilon^2 y_3 A_{31} - \varepsilon^4 (P_{31} + A_{31}^2) + \ldots \tag{3.3}$$
$$v = -\varepsilon^3 \frac{\partial A_{31}}{\partial x} + \varepsilon^3 y_3^2 \frac{\partial A_{31}}{\partial x} - \varepsilon^5 \frac{\partial A_{32}}{\partial x} + \ldots$$
$$p = \varepsilon^4 P_{31} + \varepsilon^4 y_3 \frac{\partial^2 A_{31}}{\partial x^2} + \varepsilon^6 P_{32} + \ldots$$

Although it is possible to satisfy the conditions at the free surface (1.2), (1.3) using (3.3), this will lead instantly to the corollary $\eta \sim \varepsilon^2$, i.e. conditions will hold for y_3 such that the last three terms of the function u will be of the same order. We shall satisfy conditions (1.2) and (1.3) more accurately using the method of matching the asymptotic expansions. With this in mind we introduce a new region 2 with characteristic transverse dimension ε^2, and choose the dependence of the functions sought on ε so as to ensure matching at the level of the exponents with (3.3)

$$u = 1 + \varepsilon^4 u_{21} + \ldots, \quad v = \varepsilon^3 v_{21} + \ldots \tag{3.4}$$
$$p = \varepsilon^4 p_{21} + \ldots, \quad \eta = \varepsilon^2 \eta_{21}(t, x) + \ldots, \quad y_3 = \varepsilon^2 y_2$$

Complete matching as $y_2 \to \infty$ at the level of the functions, yields

$$u_{21} \to -y_2^2 - 2y_2 A_{31} - A_{31}^2 - P_{31}, \quad v_{21} \to -\frac{\partial A_{31}}{\partial x}, \quad p_{21} \to P_{31} \tag{3.5}$$

Let us substitute the expansions (3.4) into the system of Navier-Stokes equations. We obtain

$$\frac{\partial v_{21}}{\partial y_2} = 0, \quad \frac{\partial u_{21}}{\partial x} + v_{21} \frac{\partial u_{21}}{\partial y_2} = -\frac{\partial p_{21}}{\partial x}, \quad \frac{\partial p_{21}}{\partial y_2} = 0 \tag{3.6}$$

The solution of system (3.6) satisfying condition (3.5) is easily obtained

$$u_{21} = -y_2^2 - 2y_2 A_{31} - A_{31}^2 - P_{31}, \quad v_{21} = -\frac{\partial A_{31}}{\partial x}, \quad p_{21} = P_{31} \tag{3.7}$$

and it is fully identical with its limiting expression (3.5). Thus the introduction of a new region 2 did not yield a new result. It was, however, necessary to carry out an analysis in this region, due to the differences in the systems of Eqs.(1.9), (1.10) and (3.6).

Using (3.7) and satisfying conditions (1.2) and (1.3) we obtain

$$\eta_{21} = -A_{31}, \quad P_{31} = -S_T \partial^2 A_{31}/\partial x^2 \tag{3.8}$$

The last equation in (3.8), which defines the relations connecting A_{31} with P_{31}, provides the closure of the problem for region 4 adjacent to the wall.

4. The system of Eqs.(3.6) contains no second derivatives in y_2. Therefore we could satisfy the conditions at the surface and simultaneously establish a relation between P_{31} and A_{31} only because condition (1.3) was found to follow, within the approximation used, from the first condition of (1.2) and the analytical form of solution (3.7). In order to retain the second derivative in the defining equations, we shall introduce a new region 1 and approach the free surface even more closely. Let us write the expansions in the new region in greater detail

$$u = 1 + \varepsilon^4 u_{11} + \varepsilon^6 u_{12} + \ldots, \quad v = \varepsilon^3 v_{11} + \varepsilon^5 v_{12} + \ldots \tag{4.1}$$
$$p = \varepsilon^4 p_{11} + \varepsilon^6 p_{12} + \ldots, \quad y = -\varepsilon^2 A_{31} + \varepsilon^3 y_1, \quad \eta = -\varepsilon^2 A_{31} + \varepsilon^3 \eta_{12} + \varepsilon^4 \eta_{13} + \ldots$$

Substituting expansions (4.1) into the Navier-Stokes equations, we arrive at the system

$$\frac{\partial v_{11}}{\partial y_1} = 0, \quad \frac{\partial p_{11}}{\partial y_1} = 0, \quad \frac{\partial u_{11}}{\partial x} = -\frac{\partial p_{11}}{\partial x} + \frac{2}{3} \frac{\partial^2 u_{11}}{\partial y_1^2} \tag{4.2}$$

Its solution, which satisfies the limiting conditions

$$u_{11} \to -P_{31}, \quad v_{11} \to -\partial A_{31}/\partial x, \quad p_{11} \to P_{31} \tag{4.3}$$

as $y_1 \to \infty$ and the conditions at the free surface, is unique and is the same as these limiting conditions.

Although a second derivative in y_1 appears in Eqs.(4.2), the limiting conditions and conditions at the free surface prevent us from obtaining a solution different from the limiting solution (4.3).

We obtain the following inhomogeneous system for the functions with the subscript 12:

$$\partial v_{12}/\partial y_1 = 0, \quad \partial p_{12}/\partial y_1 = 0 \tag{4.4}$$

$$\frac{\partial u_{12}}{\partial x} + \frac{\partial p_{12}}{\partial x} - \frac{2}{3}\frac{\partial^2 u_{12}}{\partial y_1^2} = -\frac{\partial u_{11}}{\partial t} + \frac{4}{3}$$

Complete formulation of the problem for system (4.4) is possible only after constructing the higher-order terms in expansions (3.4). Without carrying out bulky derivations, we shall write out the most important result $\eta_{12} = 0$.

An analysis carried out for the functions in higher approximations, makes it possible to determine the function η_{13} and to find the relation connecting the functions A_{32} and P_{32} introduced in (3.2)

$$P_{32} = -S_T\frac{\partial^2 A_{32}}{\partial x^2} + S_T\frac{\partial^2 A_{31}}{\partial t\partial x} + A_{31}\frac{\partial^2 A_{31}}{\partial x^2} \tag{4.5}$$

The relation connecting A_{32} and P_{32} enables us to formulate the problem for region 4 in the next higher approximation.

This shows that introducing a new region 1 leads to equations whose order corresponds to the number of conditions at the free surface.

5. The analysis carried out above enabled us to study the special features of one of the possible forms of asymptotic solutions when $R \to \infty$. The flow in transverse direction was found to be divided into four distinct regions. In the first three regions we have essentially succeeded in constructing the solution in explicit form, and this was mainly possible because the defining equations were linear. In the lower most region 4 however, the system of defining Eqs.(2.6) was found to be non-linear, and its solution had therefore to be obtained numerically. The conditions of matching (2.4) together with the second equation of (3.8) reflecting the influence of the free surface, enabled us to close the problem for system (2.6) and to consider the solution for region 4 only. The solution of this problem gave the function A_{31}, and through it the exact form of solutions for all upper regions.

Two characteristic groups of problems for system (2.6) should be singled out. The first group will contain the problems in which the initial conditions (2.9) and $t = 0$ and conditions at the wall (2.8) at $t > 0$ are given. Choosing various combinations of the functions u_{410}, u_w, v_w, we can solve physical problems which have straightforward analogues with the problems of a boundary layer with selfinduced pressure on a flat plate. Examples of such problems include the problems of injection or suction, the development of vortex formations, and the harmonic vibrator and of its starting /12/. In all problems of the first group, condition (3.8) holds for all x, and no forcing conditions are specified at the surface of separation, i.e. at the free surface.

The second group will include the problems in which perturbing forces are applied to the surface of separation $y' = \eta$. For example, in /6/ an external pressure was applied to part of this surface which was no longer free. In such problems we must relinquish the second condition of (1.2) on this part of the surface, and this leads to relinquishing the relation (3.8) and replacing it by

$$P_{31} = -S_T\frac{\partial^2 A_{31}}{\partial x^2} + P_S$$

where $P_S(t, x)$ is the external pressure.

Let us turn our attention to region 4 and make an additional simplifying assumption which will enable us to linearize the problem with respect to the unperturbed flow. Let the functions with the subscript 41 have the form

$$u_{41} = 2y_4 + \delta u_{41}' + \ldots, \quad v_{41} = \delta v_{41}' + \ldots \tag{5.1}$$

$$p_{41} = \delta p_{41}' + \ldots, \quad \delta \ll 1$$

The system for the functions with a prime follows from (2.6) and (5.1), and has the form

$$\partial u_{41}'/\partial x - \partial v_{41}'/\partial y_4 = 0, \quad \partial p_{41}'/\partial y_4 = 0 \tag{5.2}$$

$$\frac{\partial u_{41}'}{\partial t} + 2y_4\frac{\partial u_{41}'}{\partial x} - 2v_{41}' = -\frac{\partial p_{41}'}{\partial x} + \frac{2}{3}\frac{\partial^2 u_{41}'}{\partial y_4^2}$$

The limiting conditions (2.4) and (3.8) yield

$$p_{41}' = {}^1/_2\,(S_T - {}^8/_{15})\,(\partial^2 u_{41}'/\partial x^2)_{y_4\to\infty} \tag{5.3}$$

As we said before, the boundary conditions at the surface (2.8) and initial conditions (2.9) enable us to consider various physical problems. We will dwell on the simplest problem of natural oscillations, putting

$$u_{41w} = 0, \quad v_{41w} = 0 \tag{5.4}$$

Following the methods of the theory of stability, we shall seek the solution of problem (5.2)-(5.4) in the form

$$u_{41}' = -e^{ikx+i\omega t} df(y_4)/dy_4, \quad v_{41}' = e^{ikx+i\omega t} ikf(y_4) \tag{5.5}$$
$$p_{41}' = e^{ikx+i\omega t}$$

Substituting (5.5) into system (5.2) we obtain an equation /4/ which can be reduced, after differentiating with respect to y_4 and introducing a new independent variable $z = -(3ik)^{1/3} y_4 + 3i\omega (3ik)^{-2/3}/2$, to the Airy equation for the function d^2f/dz^2.

The solution of the last equation satisfying the condition that df/dz has a limit as $y_4 \to \infty$, has the following form for $^1/_2\pi > \arg k > -^3/_2\pi$:

$$d^2f/dz^2 = B \operatorname{Ai}(z)$$

where $\operatorname{Ai}(z)$ is the Airy function. Satisfying the conditions at the point $y_4 = 0$, as $y_4 \to \infty$ we find that the constant B will differ from zero only in the case when the wave number k and the frequency ω are connected by the following relation:

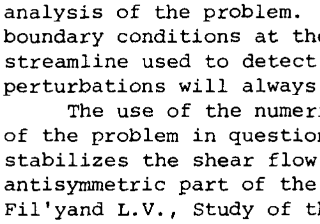
$$\frac{d \operatorname{Ai}(\Omega)}{dz} \left[\int_{\Omega}^{\infty} \operatorname{Ai}(z)\, dz \right]^{-1} = \pm \frac{k_1^2}{60} \left(\frac{1}{2} ik_1 \right)^{1/3} \tag{5.6}$$

$$\Omega = i\omega_1 (^1/_2 ik_1)^{-2/3}, \quad k = A_0 k_1, \quad \omega = B_0 \omega_1$$
$$A_0 = 15^{-3/7} 6^{-1/7} | S_T - ^8/_{15} |^{-3/7}, \quad B_0 = (^2/_3)^{1/3} A_0^{2/3}$$

The plus sign should be taken when $S_T < ^8/_{15}$ and the minus sign when $S_T > ^8/_{15}$. Eq.(5.6) with plus sign connects the wave number with the frequency of natural oscillations in plane Poiseuille flow whose behaviour has been studied in detail /11/. Using the results of /11/ we can conclude that when the value of $k \neq 0$ is fixed, Eq.(5.6) has a denumerable set of roots $\omega_{1,n}(k)$. When k is varied continuously, the roots $\omega_{1,n}(k)$ form a denumerable set of trajectories. If we choose k so that it varies along the positive part of the real axis, then all roots $\omega_{1,n}(k)$, beginning with the second root, will lie in the second quadrant of the k-plane. Only the first root $\omega_{1,1}(k)$ will move from the second quadrant into the third, intersecting the real axis at the point $\omega_{1,1}(k) = \omega_* = -4.981$ with $k_* = 6.385$. The values of k_* and ω_* will determine the neutral oscillations, the perturbations will decay for $k < k_*$ with time, and for $k > k_*$ they will increase exponentially in proportion to $\exp(-t \operatorname{Im} \omega_{1,1}(k))$. Using the transformations (1.6) with parameters (2.7) and (5.6), we can write the asymptotic expression for the neutral stability curve

$$k_{*0} = 1.549 R^{-1/7} H_0^{-1} (^8/_{15} - S_T)^{-3/7}, \quad ^8/_{15} < S_T$$
$$\omega_{*0} = 0.2119 R^{1/14} (g_0/H_0)^{1/2} \sin^{1/2}\theta \, (^8/_{15} - S_T)^{-2/7}$$

Analysing the roots of Eq.(5.6) in the case when ω_1 varies along the real axis /11/, we can show the root $\omega_1 = 0$, $k_1 = -5.728$, specifying the asymptotic solution which passes, at the non-linear stage, into a stationary separation.



The figure shows the relationship $\omega_{1,n}(k)$ for the case when k varies along the real axis, for

$$S_T > ^8/_{15} \tag{5.7}$$

Since not a single trajectory intersects the real axis, it follows that the flows obeying the condition (5.7) are stable. In this case the roots will not include the root specifying the asymptotic forms of the stationary separation.

Clearly, when R increases, according to (1.4), S_T will decrease and in the limit, as $R \to \infty$, the maintenance of the inequality (5.7) should rather be treated as a formal parametric analysis of the problem. It is interesting to note that in the limit, as $S_T \to \infty$, the boundary conditions at the free surface are identical with the conditions on the central streamline used to detect the antisymmetric perturbations in plane Poiseuille flow. Such perturbations will always be stable /13/.

The use of the numerical solution of the Orr-Sommerfeld equation in the study of stability of the problem in question shows that increasing the parameter S_T, from some specified value stabilizes the shear flow and brings the sepctrum of the perturbed problem closer to the antisymmetric part of the spectrum of perturbed Poiseuille* (*Belikov V.V., Epikhin V.E. and Fil'yand L.V., Study of the stability of flows with a surface of separation (capillary jets, liquid layers). Report of NIImekhaniki MGU, 2450, 1980.) flow. The asymptotic analysis carried out shows, that an analogous dependence on the parameter S_T is also observed in the limiting perturbed flow as $R \to \infty$.

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PROBLEMS OF THE INTERACTION OF A BLUNT BODY WITH AN ACOUSTIC MEDIUM*

F.M. BORODICH

The initial (supersonic) stage of the interaction of a blunt body (penetration and impact) with an acoustic medium (a compressible fluid) is examined in a laminar formulation. It is assumed that the boundary of the domain of interaction of the body with a medium moves at a velocity exceeding the velocity of sound in the medium. Explicit formulas are derived for the velocity of the particles of the medium and the pressure at each point of the interaction domain boundary. It is shown that the general solution of the linearized problem for the supersonic stage of blunt body penetration, given by an explicit formula /1-3/ in the form of a double integral, can be converted in such a manner as to reduce the formula to a single integral for an arbitrary body penetrating the fluid at an arbitrary velocity. Earlier only problems of the penetration of bodies of revolution bounded by second-order surfaces (cone /3, 4/, paraboloid /4, 5/, ellipsoid and hyperboloid /4/) at a constant velocity were investigated analytically using such a reduction. An exact expression is obtained for the law of motion on the inertia of a body of arbitrary shape after its contact with the fluid.

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